2001

On the nonembeddability and crossing numbers of some toroidal graphs on the Klein bottle

Adrian Riskin

ELSEVIER

Discrete Mathematics 234 (2001) 77–88

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

On the nonembeddability and crossing numbers of some toroidal graphs on the Klein bottle

Adrian Riskin

Department of Mathematics, Mary Baldwin College, Staunton, VA 24401, USA

Received 7 October 1997; revised 1 October 1999; accepted 10 April 2000

Abstract

We show that toroidal polyhedral maps with four or more disjoint homotopic noncontractible circuits are not embeddable on the projective plane and that toroidal polyhedral maps with five or more disjoint homotopic noncontractible circuits are not embeddable on the Klein bottle. We also show that the Klein bottle crossing numbers of $C_m \times C_n$ $(m \leqslant n)$ for $m = 3, 4, 5, 6$ are 1,2,4, and 6, respectively, and give upper bounds for all other values of n. These crossing numbers display atypical behavior in that the value depends only on m instead of on both m and n as is the case for the plane and projective plane.

Keywords: Crossing numbers; Polyhedral maps; Representativity; Reimbedding

1. Introduction

It is a well known yet still amusing fact that K_7, while embeddable in the torus, is not embeddable in the Klein bottle [3]. Since the two surfaces are so similar, this situation would seem to be a fairly special one. We show that in some sense it is not so rare by giving a broad infinite class of toroidal graphs, i.e. toroidal polyhedral maps with five or more disjoint homotopic noncontractible circuits, which are not embeddable in the Klein bottle (a map is *polyhedral* if no two closed faces have a multiply connected union). We also show that toroidal polyhedral maps with four or more disjoint homotopic noncontractible circuits are not projective planar. The *representativity* of an embedding of a graph in a surface S the minimum number of points in which a noncontractible curve in S meets the graph; thus a polyhedral embedding has representativity at least 3. Our results concerning nonembeddability are in the spirit of a number of interesting theorems about representativity and reimbeddings by Robertson, Vitray, Archdeacon, and others (see [13] for a useful survey).

The plane crossing numbers of the graphs $C_m \times C_n$ were first studied by Harary et al. [4]. Note that when considering $C_m \times C_n$ we always assume $m \leqslant n$. The exact values

E-mail address: ariskin@mbc.edu (A. Riskin).

0012-365X/01/$ - see front matter
PII: S0012-365X(00)00193-X

for $m=3,4$, and 5 are [1,10,8] n, $2n$, and $3n$, respectively. The author [11] calculated the projective plane crossing numbers of $C_3 \times C_n$ to be 2 for $n=4$ and $n-1$ for $n>4$. Recently, a great deal of important work has been done on the plane crossing numbers of these graphs by Richter and Thomassen [9] and Klesc et al. [8]. In this paper, we show that the Klein bottle crossing numbers of $C_m \times C_n$ for $m=3,4,5$, and 6 are 1, 2, 4, and 6, respectively. We also give an upper bound for the Klein bottle crossing numbers of $C_m \times C_n$ for $7 \leqslant m$ and conjecture that equality holds. Note that it follows from the results of [1,10,11] that the crossing number of $C_m \times C_n$ increases with both m and n on both the plane and projective plane. Interestingly, this turns out not to be the case for the Klein bottle. It follows from our results that the Klein bottle crossing number of $C_m \times C_n$ depends only on m for $3 \leqslant m \leqslant 6$. This is also the case for arbitrary m and sufficiently large n, and we conjecture it is always the case.

2. Definitions and preliminary material

A *map* is an embedded graph. We denote the underlying graph of a map M by $G(M)$ when it is necessary to distinguish between them. We tend to conflate the two concepts when no confusion is likely to arise, as well as to conflate faces of a map with their closures or with their bounding circuits in the underlying graph. If H is a subgraph of G, we denote the subgraph obtained by removing the vertices of H by G–H. A circuit C in a graph G is *peripheral* provided that G–C is connected and C has no chords.

If C is a circuit in a graph G, then the *C-contraction* of G, denoted by G/C, is the graph obtained from G by contracting C to a vertex. If C is a circuit in the underlying graph $G(M)$ of a map M in the (possibly pseudo) surface S which is noncontractible with respect to the embedding M, then the *C-contraction* of M is either the induced embedding of $G(M)/C$ in the (possibly pseudo) surface S/C obtained by collapsing C in S to a single point, or else that (possibly pseudo) surface itself. Also, a *polyhedral subannulus* of a toroidal polyhedral map M is a noncontractible subcomplex of M which is homeomorphic to an annulus, and which becomes a spherical polyhedral map when the two bounding circuits are capped. The *relative interior* of a polyhedral subannulus S, denoted *relint* S, is S without its bounding disjoint homotopic noncontractible circuits. The standard toroidal embedding of $C_m \times C_n$ is the natural self-dual toroidal quadrangulation.

Let P be a simple path in a map M. If x and y are vertices of P, then $P[x,y]$ is the subpath of P joining x and y. P is called a *W_v-path* if for each face F and vertices $x,y \in P \cap F$, $P[x,y] \subset F$. This terminology is due to Klee [6,7], who used a 'W' in honour of the creator, Wolf [David Barnette; personal communication]. A *revisit* of P is a pair of vertices $\{x,y\} = P[x,y] \subset F$ for some face F of M. We say P *revisits* F if this occurs. A simple circuit in M is called a *W_v-circuit* if for each pair of vertices $x,y \in C$, $\{x,y\}$ is a revisit for at most one of the two xy-paths along C. Finally, we denote the crossing number of G on the Klein bottle by $\overline{\overline{cr_2}}(G)$. All other terms are standard, and may be found, e.g. in [2].

3. On the nonembeddability of toroidal maps

Lemma 3.1. *Let M be a toroidal polyhedral map with k disjoint homotopic noncontractible circuits $C_1 \cdots C_k$. Then M has k disjoint homotopic noncontractible circuits which partition M into k polyhedral subannuli.*

Proof. The k circuits $C_1 \cdots C_k$ partition M into k annuli, which will fail to be polyhedral only if some C_i revisits a face. Suppose without loss of generality that $C_1 \cdots C_k$ are a set of disjoint homotopic noncontractible circuits which yield the lowest total number of such revisits. If they yield no revisits, we are done, so assume that, e.g., C_1 revisits a face F. Let x and y be two vertices in different components of the revisit. Then replacing the appropriate $C_1[x, y]$ by one of the two paths along F from x to y will eliminate that particular revisit. Since M is polyhedral, no new revisits are introduced, and it is clear that an appropriate choice of the xy-path along F will prevent the modified C_1 from intersecting any of the other C_i's. Thus, the modified C_1, along with $C_2 \cdots C_k$ are a set of k disjoint homotopic noncontractible circuits which have fewer total revisits, contrary to assumption. □

We will also need the following three lemmas of Klee [6,7], Tutte [14], and the author [12], respectively.

Lemma 3.2. *Every polyhedron (i.e. plane polyhedral map) has a W_v path between each two vertices.*

Lemma 3.3. *A circuit C of a planar 3-connected graph G bounds a face in every planar embedding of G iff C is peripheral.*

Lemma 3.4. *If C is a peripheral circuit in a graph G such that its contraction yields a nonplanar graph, then C must bound a face in any projective plane embedding of G.*

We will also use the following evident facts frequently and without special mention: A plane graph is polyhedrally embedded if and only if it is 3-connected; collapsing a noncontractible curve (or circuit of an embedded graph) on the projective plane yields the sphere; and a peripheral circuit of a graph embedded on a surface must either bound a face or be noncontractible. We are now in a position to prove:

Theorem 3.5. *No toroidal polyhedral map with four disjoint homotopic noncontractible circuits is embeddable in the projective plane.*

Proof. Suppose that M is such a map on the torus. By Lemma 3.1, we may assume that the four disjoint homotopic noncontractible circuits C_1, C_2, C_3, and C_4 bound polyhedral subannuli S_1, S_2, S_3, and S_4, where S_i is bounded by C_i and C_{i+1} (subscripts taken mod 4). Let F be a face of M, and assume $F \subset S_1$. If S_1 is contracted to a point, what remains is a polyhedral map M^* on the pseudosurface known as the pinched torus (i.e. a sphere with two points identified). Since M^* is polyhedral, every face is bounded by a peripheral circuit. Suppose that the graph G of M^* is planar. Then by Lemma 3.3, every facial circuit of M^* bounds a face in the plane embedding of G. Since the Euler characteristic of the pinched torus is 1, which is one less than that of the plane, this is a violation of Euler's theorem. Since M^* is a subcontraction of the graph obtained from M by contracting F to a point, that graph must be nonplanar as well.

Thus by Lemma 3.4, F and, mutatis mutandis, any face of M, must be a face in any projective plane embedding of M. Since the Euler characteristic of the projective plane is one greater than that of the torus, any projective plane embedding of the graph of M must have one face more than M. However, that will necessitate 3 faces containing some edge, which is impossible. □

Lemma 3.6. *The only possible contractions of the Klein bottle are the pinched torus, the projective plane, and the space obtained by joining two projective planes at a point.*

Proof. Let C be a noncontractible curve in the Klein bottle. We now consider the torus as the smooth orientable double cover of the Klein bottle. The preimage B of C in the torus is either a single noncontractible curve or a pair of disjoint homotopic noncontractible curves. In the first case, the B-contraction of the torus is the pinched torus. When the C-contraction of the Klein bottle is re-covered by the B-contraction of the torus, the other pairs of antipodal points are identified as well and the result is the projective plane.

In the second case, collapsing B yields a pair of spheres joined to one another at two points (a 'two pearl necklace'). Here when the C-contraction of the Klein bottle is re-covered with the B-contraction of the torus, either the antipodal points of each sphere are separately identified, in which case the result is two projective planes joined at a point, or else the points in one sphere are identified with the points in the other sphere, in which case the result is the pinched torus. □

Note that if the collapsed noncontractible curve C happens to be a peripheral circuit of a map on the Klein bottle, then the resulting C-contraction can only be the pinched torus or the projective plane. This is because any circuit in a 2-cell embedded graph whose contraction produces the two joined projective planes must separate the graph.

Lemma 3.7. *Let M be a toroidal polyhedral map with five or more disjoint homotopic noncontractible circuits, and let F be a face of M. Removing the bounding circuit of F from $G(M)$ yields a nonplanar graph.*

Proof. Let $C_1, \ldots, C_5$ be five disjoint homotopic noncontractible circuits of M. By Lemma 3.1 they can be assumed to bound polyhedral subannuli $P_1, \ldots, P_5$ of M. We assume further that these are labelled in such a way that P_i is bounded by C_i and C_{i+1} (subscripts taken mod 5). Suppose without loss of generality that $F \subseteq P_3$. Now, let u be a vertex in C_2 and v a vertex in C_5. Note that the disjoint homotopic noncontractible circuits C_2 and C_5 induce a decomposition of M into two plane polyhedral maps P and Q, where we suppose that $F \subseteq Q$. Since F bounds a face in the unique (by Lemma 3.3) plane embedding of Q, F is peripheral in Q. Thus, there is a path H in Q from u to v which misses F. The existence of C_1 insures that u and v are not in the same face of the unique plane embedding of P. Thus, $P \cup H$ is nonplanar, and since $P \cup H \subseteq G(M) - F$, $G(M) - F$ is nonplanar as well. □

Lemma 3.8. *Let M be a toroidal polyhedral map with five or more disjoint homotopic noncontractible circuits, and let F be a face of M. Then if $G(M)$ has a Klein bottle embedding in which F does not bound a face, the F-contraction of the Klein bottle must be the projective plane.*

Proof. Since F bounds a face in polyhedral map M, it is peripheral in $G(M)$, and so the F-contraction cannot be the two joined projective planes. Furthermore, by Lemma 3.7, $G(M) - F$ is nonplanar, and so the F-contraction cannot be the pinched torus. Thus, by Lemma 3.6, the F-contraction must be the projective plane. □

We are now able to prove our:

Theorem 3.9. *No toroidal polyhedral map with five or more disjoint homotopic noncontractible circuits is embeddable in the Klein bottle.*

Proof. By way of contradiction, suppose that M is such a toroidal polyhedral map which has a Klein bottle embedding K. We will obtain a contradiction from this supposition by showing that every face of M is a face of K (and thus by Euler's theorem vice versa), which, once shown, implies the (contradictory) existence of a patchwise defined homeomorphism between the torus and the Klein bottle. Let F be a face of M. By way of contradiction, assume F does not bound a face of K. As in the proof of Lemma 3.7, we assume that the five disjoint homotopic noncontractible circuits $C_1, \ldots, C_5$ bound polyhedral subannuli $P_1, \ldots, P_5$ labelled as above, with $F \subseteq P_3$. Now, by Lemma 3.8, K/F is the projective plane. Let e be an edge of C_5 in $G(M)/F$, and let H_1 and H_2 be the two faces containing e in M. Note that $H_1 \cap C_4$ may or may not be empty.

Now, $K_3 \times K_2$ is a subcontraction of P_1 and K_4 is a subcontraction of P_5/C_5, so the amalgamation of the two graphs along a triangle (Fig. 1) is a subcontraction of $(P_1 \cup P_5)/C_5$. Clearly, there is a path in $P_2 \cup P_3 \cup P_1$ joining C_5 to C_2, and, thus, from vertex x to C_1 in $(P_2 \cup P_3 \cup P_4)/C_5$. Thus $(G(M)/F)/C_5$ is nonplanar, and since C_5 is peripheral in $G(M)/F$, C_5 must bound a face in the induced projective plane embedding of $G(M)/F$. Similar arguments show that H_1 and H_2 must also bound faces

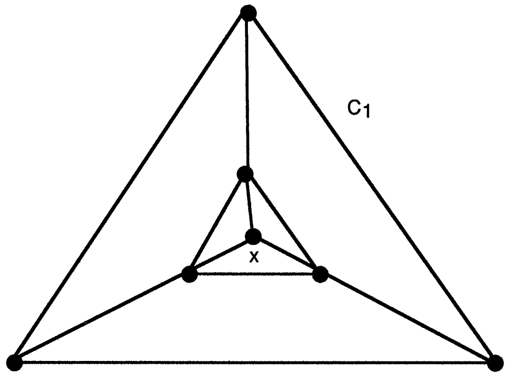

Fig. 1.

in that embedding, yielding three faces on edge e in the embedding, a contradiction. Thus, F must be a face in K. □

Lemma 3.10. *$C_4 \times C_4$ is not embeddable on the Klein bottle.*

Proof. $C_4 \times C_4$ is the 4-cube, which was shown by Jungerman [5] to have nonorientable genus 3. □

Corollary 3.11. *$C_3 \times C_3$ and $C_3 \times C_4$ are the only products of cycles which are embeddable in the Klein bottle.*

4. Klein bottle crossing numbers of products of cycles

Theorem 4.1. *Except for $m = 3$ and $n = 3$ or 4*

$$\overline{\overline{cr_2}}(C_m \times C_n) = \begin{cases} 2t_{k-1} = k^2 - k & \text{for } m = 2k, \\ t_{k-1} + t_k = k^2 & \text{for } m = 2k+1, \end{cases}$$

where $m \leqslant n$ and t_k is the kth triangular number $k(k+1)/2$.

Proof. Figs. 2–4 show drawings of $C_3 \times C_n$, $C_4 \times C_n$, and $C_5 \times C_n$ with 1, 2, and 4 crossings, respectively. Note that in the figures the vertex labels show how the boundaries are to be identified. Note that the crossings in the drawing of $C_{2k} \times C_n$ consist of two triangular groups, each with $(k^2 - k)/2$ crossings. To obtain the drawings of $C_{2k+1} \times C_n$ from that of $C_{2k} \times C_n$, the new meridian is drawn across each of the edges involved in one of these groups of crossings, adding k new crossings, and yielding $k^2 - k + k = k^2$ crossings altogether. The drawing of $C_{2k+2} \times C_n$ is obtained from that of $C_{2k+1} \times C_n$ in a similar manner. □

Corollary 4.5. $\overline{\overline{cr_2}}(C_3 \times C_n) = 1$ *for* $n \geqslant 5$.

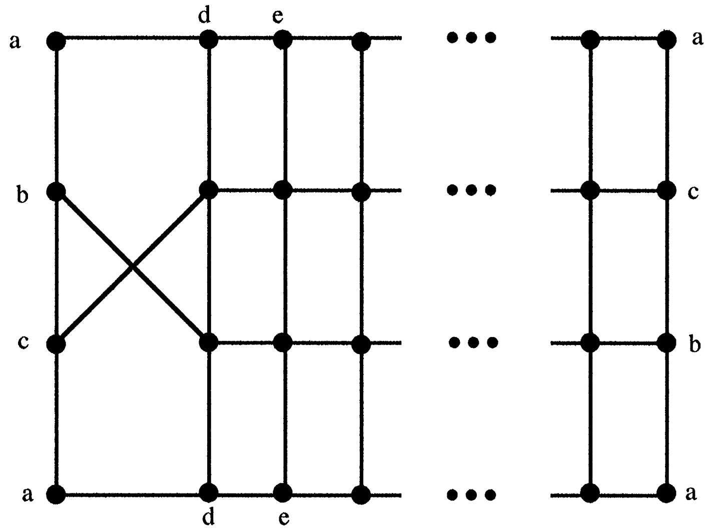

Fig. 2.

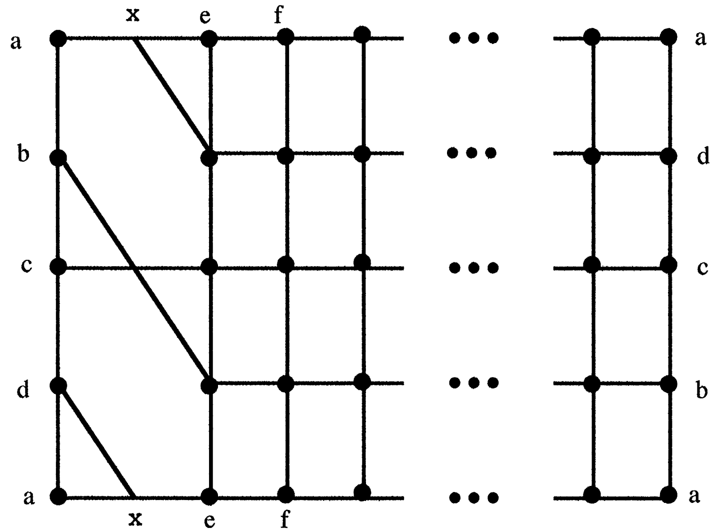

Fig. 3.

Proof. Apply Theorem 4.1 and Corollary 3.10. □

Now, in order to establish our result for $m = 4$, we will need two lemmas (the first is from [12]).

Lemma 4.6. *If a polyhedral map M on the pinched torus has three disjoint circuits which bound disks containing the pinch point, then* $G(M)$ *is not projective planar.*

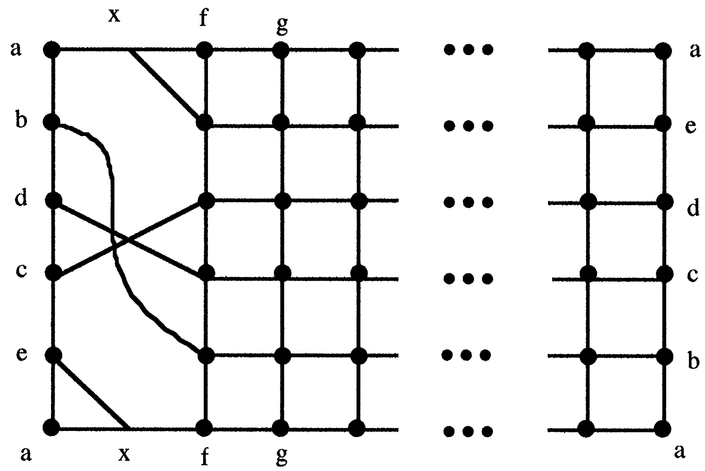

Fig. 4.

Lemma 4.7. *If $C_3 \times C_4$ is embedded in the Klein bottle, then all four triangles must bound faces.*

Proof. Since the four triangles are peripheral circuits, they must either bound faces or else be noncontractible. If one triangle, say T, is noncontractible, then the T-contraction graph G has an induced imbedding in one of the three contractions of the Klein bottle. Since G is not merely 1-connected, the contraction consisting of the two joined projective planes is not possible. Furthermore, because the T-contraction of $C_3 \times C_4$ is exactly the type of map to which Lemma 4.6 applies, the projective plane is also ruled out as a possible contraction. Thus the only possibility is the pinched torus. In the induced embedding the vertex v to which T was collapsed lies on the pinch point. Thus, removing v and separating the pinched torus at the pinch point yields $K_2 \times C_3$ embedded in the sphere. The graph $K_2 \times C_3$ is planar and 3-connected, so by Steinitz's Theorem has exactly one embedding in the sphere. Thus, the T-contraction has exactly one embedding on the pinched torus. However, it is clear that this embedding cannot come from collapsing T on the Klein bottle, but only from collapsing T on the torus, a contradiction. Thus no triangle of $C_3 \times C_4$ is noncontractible in the embedding, so their nonseparability implies that they all bound faces. □

Theorem 4.8. $\overline{\overline{cr_2}}(C_4 \times C_n) = 2$.

Proof. In this proof, it will be convenient to denote the length of the bounding circuit of a face F by $\pounds(F)$. Clearly it will suffice to show that $\overline{\overline{cr_2}}(C_4 \times C_4) = 2$. By way of contradiction, assume there is a drawing of $C_4 \times C_4$ on the Klein bottle with only one crossing. Then $(C_4 \times C_4)$-e is embeddable on the Klein bottle. This embedding in turn induces an embedding M of $C_3 \times C_4$ on the Klein bottle which has a face F

which contains three corresponding edges from three of the triangles of $C_3 \times C_4$ (this last because the embedding is induced by an embedding of $(C_4 \times C_4)$-e). Since the triangles of $C_3 \times C_4$ are disjoint, $\pounds(F) \geqslant 6$.

However, there are no appropriate circuits in $C_3 \times C_4$ of length 6, and standard counting arguments imply that each face in a Klein bottle embedding of $C_3 \times C_4$ has length at most 8, so that $\pounds(F) = 7$ or 8. If $\pounds(F) = 7$, standard counting arguments imply that M has a face of length 5. However, each circuit in $C_3 \times C_4$ of length 5 has a chord. Thus a circuit of length 5 cannot bound a face since by Lemma 4.7, every triangle bounds a face, and two faces which meet in two consecutive edges imply the existence of a 2-valent vertex, of which there are none in $C_3 \times C_4$.

Thus $\pounds(F) = 8$. It follows from a similar argument that F contains corresponding edges from all four triangles of $C_3 \times C_4$, and that these occur in a cyclic order which will admit completion up to a Klein bottle embedding of $C_4 \times C_4$. This contradicts Corollary 3.10. □

Finally, in order to establish our result for $m = 5$ and 6, we need the following lemmas:

Lemma 4.9. *If D is a* 3 *crossing drawing of $C_5 \times C_5$ on the Klein bottle, then* (a) *no edge is crossed twice in D, and* (b) *no vertex is incident with two crossed edges in D.*

Proof of (b). If there were such a vertex in D, removing it would yield a 1 crossing drawing of a graph of which $C_4 \times C_4$ is a subgraph, contradicting Theorem 4.6. Part (a) follows directly from (b). □

Lemma 4.10. *If D is a* 3*-crossing drawing of $C_5 \times C_5$ on the Klein bottle, the six crossed edges of D must lie on parallel meridians.*

Proof. By way of contradiction, assume that one crossed edge lies in a meridian orthogonal to a meridian containing another crossed edge. Then by Lemma 4.9 there will exist two crossed edges in orthogonal meridians which do not cross one another. Removing these two meridians yields a 1-crossing drawing of $C_4 \times C_4$, in contradiction to Theorem 4.8. □

Lemma 4.11. *If D is a* 3*-crossing drawing of $C_5 \times C_5$ on the Klein bottle and edges e and f cross one another in D, then e and f lie in the same meridian.*

Proof. Suppose by way of contradiction that there is a 3-crossing drawing D in which edges e and f cross one another and yet lie in different parallel (by Lemma 4.10) meridians M_1 and M_2. First, we will show that collapsing either M_1 or M_2 in D yields the pinched torus. First of all, both must be noncontractible in D, since if one bounded, the fact that it was once crossed would imply that it was twice crossed, and

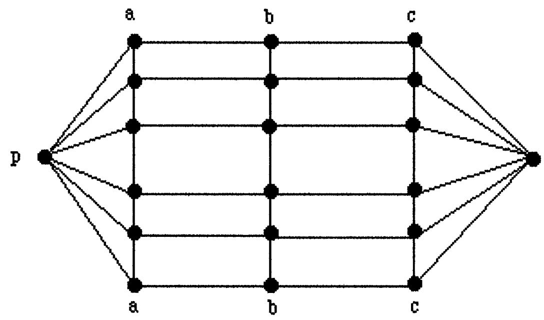

Fig. 5.

then removing it would yield the same type of contradiction obtained above. Now, by way of contradiction, assume that collapsing M_1 does not yield the pinched torus. Since M_1 is peripheral, collapsing it must then yield the projective plane. Thus, removing M_2 and collapsing M_1 induces a projective plane embedding of the graph G, shown in Fig. 5 embedded in the pinched torus, with two crossings on the three circuits homotopic to the vertex lying on the pinch point in the figure. Note that in the figure, the pinched torus is produced by identifying the upper and lower arcs to produce a 'stretched sphere', and then the two points labelled 'p' are identified to produce the pinch. Clearly, these crossings can be eliminated by removing one edge each from two of the horizontal (in the figure) face bands, leaving the remaining graph embedded in the projective plane. However, by Lemma 4.6, that graph is not projective planar. Thus collapsing M_1 must yield the pinched torus. Naturally the same argument works for M_2.

Now, when we consider the torus as the smooth twofold orientable cover of the Klein bottle, the drawing D has as preimage a drawing D' of a different graph drawn in the torus. Since M_i collapses the Klein bottle to the pinched torus, each M_i must have as its preimage in D' a pair of disjoint homotopic circuits $\{M_{i1}, M_{i2}\} \to M_i$. Clearly in the torus the set $\{M_{11}, M_{12}, M_{21}, M_{22}\}$ must have either no intersections or $\geqslant 4$ intersections. Thus downstairs in the Klein bottle M_1 and M_2 have either no intersections or $\geqslant 2$ intersections. Since we assume they have at least 1, they must have at least 2. As above, this yields a contradiction. □

Theorem 4.13. $\overline{\overline{cr_2}}(C_5 \times C_n) = 4$ $(n \geqslant 5)$.

Proof. As before, it is sufficient to show that $\overline{\overline{cr_2}}(C_5 \times C_5) = 4$. By an easy corollary of Theorem 4.8 it is at least 3. By way of contradiction, assume there is a 3-crossing drawing D of $C_5 \times C_5$ on the Klein bottle. By Lemma 4.11, each crossing consists of two edges in the same meridian which cross each other. By Lemma 4.10 those three meridians are parallel to one another. Consider then the standard embedding S of $C_5 \times C_5$. Clearly the three crossings can be eliminated by removing no more than two edges per parallel band of faces in S, leaving a polyhedral map on the torus with five disjoint homotopic noncontractible circuits embedded in the Klein bottle, in contradiction to Theorem 3.9. □

Theorem 4.14. $\overline{\overline{cr_2}}\,(C_6 \times C_n) = 6\ (n \geqslant 6)$.

Proof. As above, it is sufficient to show that $\overline{\overline{cr_2}}(C_6 \times C_6) = 6$. An easy corollary to Theorem 4.13 shows that $\overline{\overline{cr}}_2(C_6 \times C_6) \geqslant 5$. Thus by way of contradiction, assume there is a 5-crossing drawing D of $C_6 \times C_6$ on the Klein bottle. As before, all crossed edges must lie in parallel meridians. As before, there can be no more than two crossed edges per meridian, and if there are two, they must cross each other. The same type of arguments used above will show that edges which cross one another must lie in the same meridian, so that, as before, the 10 crossed edges are distributed among five parallel meridians with two in each meridian and those two crossing each other. Then all five crossings can be eliminated by removing five edges with no more than three from any parallel band of faces in the standard embedding of $C_6 \times C_6$. This leaves a polyhedral map on the torus with 6 disjoint homotopic noncontractible circuits embedded in the Klein bottle, in contradiction to Theorem 3.9. □

5. Conclusion and speculations

It is not possible to extend our Theorems 3.5 and 3.9 any farther since all toroidal maps are embeddable in the nonorientable surface with crosscap number three. The reverse problem of finding conditions on polyhedral maps on the Klein bottle sufficient to forbid toroidal embeddability is also interesting, but is complicated by the existence of the three types of noncontractible curves thereon. We conjecture that Klein bottle polyhedral maps with four disjoint homotopic noncontractible circuits of the type which collapse the Klein bottle to the pinched torus are not toroidal. Finally, we note that Corollary 3.11 shows that in some sense our nonembeddability result for the Klein bottle is the best possible. The fact that $C_3 \times C_3$ is embeddable on the projective plane shows that in the same sense our nonembeddability result for the projective plane is the best possible. We have shown that equality holds in Theorem 4.1 for $3 \leqslant m \leqslant 6$. Naturally, we conjecture that it holds for $m > 6$ as well.

References

[1] L. Beineke, R. Ringeisen, On the crossing numbers of products of cycles and graphs of order 4, J. Graph Theory 4 (1980) 145–155.
[2] G. Chartrand, L. Lesniak, Graphs and Digraphs, Wadsworth and Brooks, Monterey, CA, 1994.
[3] P. Franklin, A six colour problem, J. Math. Phys. 13 (1934) 363–369.
[4] F. Harary, P. Kainen, A. Schwenk, Toroidal graphs with arbitrarily high crossing numbers, Nanta. Math. 6 (1973) 58–67.
[5] M. Jungerman, The non-orientable genus of the n-cube, Pacific J. Math. 76 (1978) 443–451.

[6] V. Klee, Paths on polyhedra I, J. SIAM 13 (1965) 946–956.
[7] V. Klee, Paths on polyhedra II, Pacific J. Math. 16 (1966) 249–262.
[8] M. Klesc, R.B. Richter, I. Stobert, The crossing number of $C_5 \times C_n$, J. Graph Theory 22 (1996) 239–243.
[9] R.B. Richter, C. Thomassen, Intersection of curve systems and the crossing number of $C_5 \times C_n$, Discrete Comput. Geom. 13 (1995) 149–159.
[10] R.D. Ringeisen, L.W. Beineke, The crossing number of $C_3 \times C_n$, J. Combin. Theory 24B (1978) 134–136.
[11] A. Riskin, The projective plane crossing numbers of $C_3 \times C_n$, J. Graph Theory 17 (1993) 683–693.
[12] A. Riskin, Projective plane embeddings of polyhedral pinched maps, Discrete Math. 126 (1994) 281–291.
[13] N. Robertson, R. Vitray, Representativity of surface embeddings, in: B. Korte, L. Lovás, H.J. Prömel, A. Schrijver (Eds.), Paths, Flows, and VLSI-Layout, Springer, Berlin, 1990, pp. 293–328.
[14] W. Tutte, How to draw a graph, Proc. London Math. Soc. Ser. 3 13 (1963) 743–768.