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The genus 2 crossing number of K_9

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Abstract

Our main result is that a 1971 conjecture due to Paul Kainen is false. Kainen's conjecture implies that the genus 2 crossing number of K_9 is 3. We disprove the conjecture by showing that the actual value is 4. The method used is a new one in the study of crossing numbers, involving proof of the impossibility of certain genus 2 embeddings of K_8.

1. Introduction

In [4], Kainen gives a lower bound for the crossing number of an arbitrary graph on an orientable surface of arbitrary genus. He notes that, in some cases, equality holds for the complete and complete bipartite graphs and conjectures conditions, described more fully below, about when equality holds for these graphs. His conjecture implies that the genus 2 crossing number of K_9 is 3, and we falsify this by showing the actual value to be 4. Note that Guy [1] has studied the plane crossing numbers of the complete graphs, and Guy et al. [3] have studied the crossing numbers of these graphs on the torus. In no case has more than a small number of values been determined.

2. Definitions and background

We denote the orientable 2-manifold of genus n by S_n. A *good* drawing of a graph G is an immersion of G into a surface which avoids the trivialities of adjacent edges crossing, edges crossing themselves, and nonadjacent edges crossing each other more than once. We also abjure the pathology of edges crossing vertices or the immersion being more than 2 to 1. The *genus n crossing number* of a graph G, denoted by $\mathrm{cr}_n(G)$, is the minimum number of crossings in a good drawing of G on S_n.

A *face* of a graph G embedded in a surface M is a connected component of $M - G$. We often intentionally confuse faces with their bounding circuits if no

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misunderstanding is likely, as in the statement of Lemma 4. A *cellular subcomplex* of a graph embedded in a surface is a set of faces, the closure of whose union is a closed 2-cell. A *nonplanar* subcomplex is a set of faces which is not a subset of any cellular subcomplex. The *star complex* of a face F is the closure of the union of all faces whose bounding circuits meet the bounding circuit of F. Two embeddings of a graph are *congruent* if the graph has an automorphism which preserves oriented face boundaries of the embeddings. We abbreviate the terms 'clockwise' and 'counterclockwise' by CW and CCW, respectively. Finally, we use the term *fragment of a rotation* to denote a list in the appropriate order of some vertices which are consecutive in the rotation around a vertex.

Kainen [4] defines

$$\delta_n(G) = q - \frac{L}{L-2}(p - 2(1-n)),$$

where G has order p, size q, and girth L. Using Euler's theorem, he shows that $\mathrm{cr}_n(G) \geqslant \delta_n(G)$. He then defines $g(G)$ to be the greatest integer t for which $\delta_t(G) \geqslant 0$ and conjectures that if G is a complete or a complete bipartite graph, and $n = g(G)$, then $\mathrm{cr}_n(G) = \delta_n(G)$. A simple calculation shows that $g(K_9) = 2$, and thus that Kainen's conjecture implies that $\mathrm{cr}_2(K_9) = \delta_2(K_9) = 3$. We falsify this by showing that in actuality $\mathrm{cr}_2(K_9) = 4$. The veracity of the conjecture remains unresolved for the complete bipartite graphs, although we strongly suspect it to be false for all but finitely many cases.

3. The genus 2 crossing number of K_9

In the first place, we note that the drawing of K_9 on S_2 with 4 crossings given in Fig. 1 proves that $\mathrm{cr}_2(K_9) \leqslant 4$. Thus in order to prove the following theorem we need only show that $\mathrm{cr}_2(K_9) \neq 3$.

Theorem 1. $\mathrm{cr}_2(K_9) = 4$.

Lemma 1. *If there is a three-crossing drawing of K_9 on S_2 then it does not contain a twice-crossed edge.*

Proof. Let D be a 3-crossing drawing of K_9 on S_2. By way of contradiction, assume there is an edge e of K_9 which is crossed twice in D. Removing edge e yields a 1-crossing drawing of $K_9 - e$. This contradicts the consequence of Kainen's above-mentioned lower bound that $\mathrm{cr}_2(K_9 - e) \geqslant 2$. □

Following Guy and Hill [2], we define the *responsibility* of a vertex to be the total number of crossings on all the edges incident with it.

Lemma 2. *An embedding of K_8 on S_2 has either one pentagonal face or two quadrilateral faces and the rest triangles.*

Proof. Let M be an embedding of K_8 on S_2, and let r_i be the number of i-sided faces, $i \geqslant 3$. Then since K_8 has 28 edges,

$$\sum_{i=3}^{\infty} ir_i = 56.$$

By Euler's theorem, M has 18 faces, so that

$$\sum_{i=3}^{\infty} r_i = 18.$$

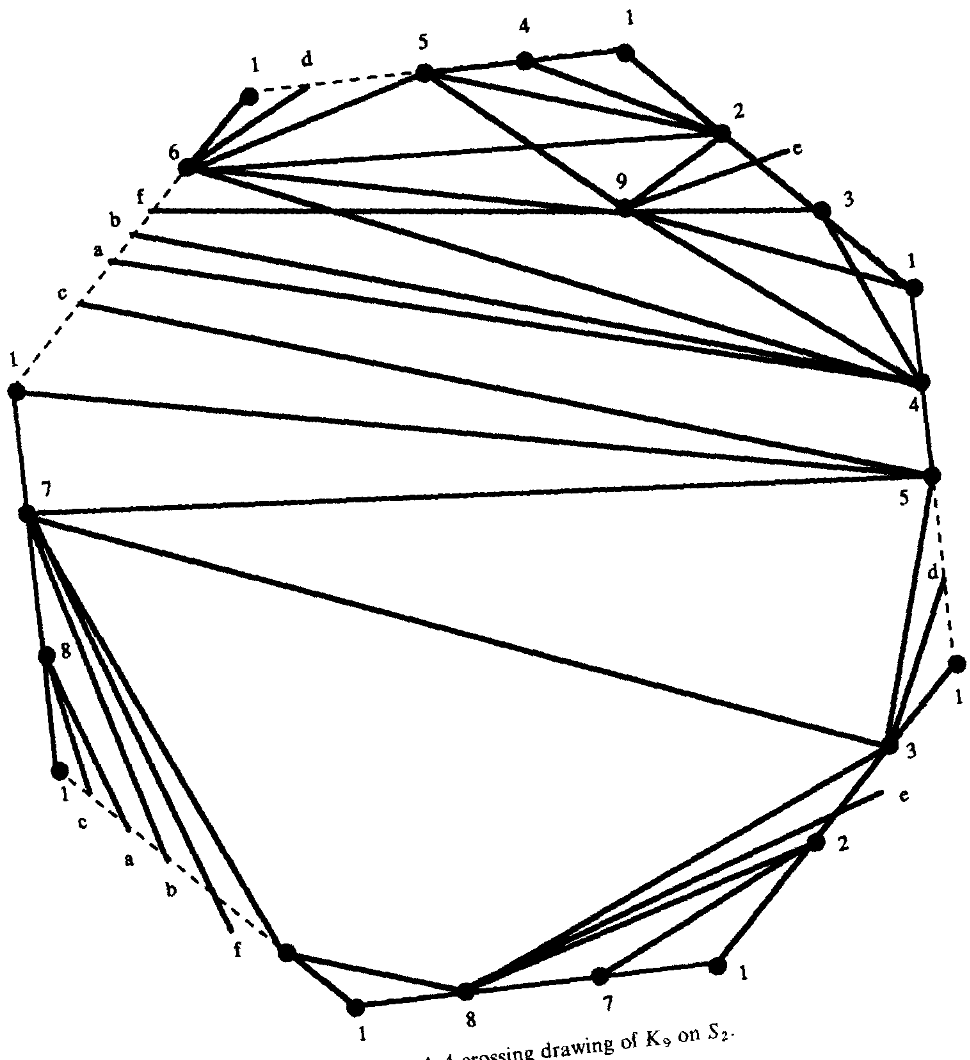

Fig. 1. A 4-crossing drawing of K_9 on S_2.

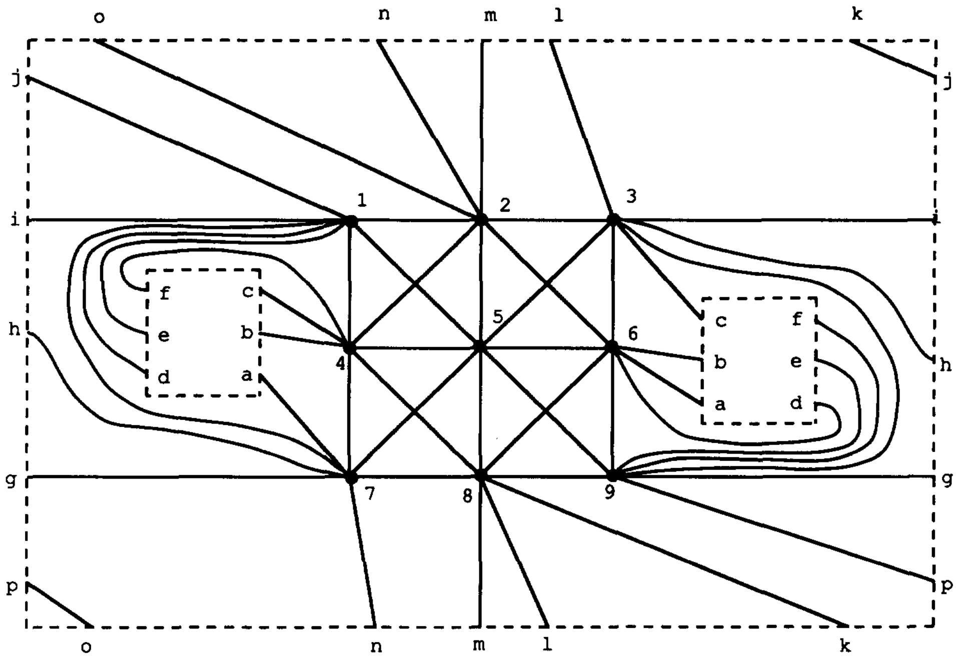

Fig. 1. *Continued.*

The only two possibilities consistent with these two equations are $r_3 = 16, r_4 = 2$, $r_i = 0$ for $i \geqslant 5$ or $r_3 = 17, r_4 = 0, r_5 = 1, r_i = 0$ for $i \geqslant 6$. $\square$

Lemma 3. *If there is a drawing of K_9 on S_2 with 3 crossings, then either there is an embedding of K_8 on S_2 with a pentagonal face or else there is an embedding of $K_8 - e$ on S_2 with a hexagonal face containing 6 distinct vertices.*

Proof. Let D be a 3-crossing drawing of K_9 on S_2. Since each crossing is in the responsibility of 4 vertices, the total responsibility of D is 12 (this argument follows Guy and Hill [2]). Thus some vertex v has responsibility either 2 or 3.

If v has responsibility 3, then removing it deletes all 3 crossings in D, and thus yields an S_2 embedding M of K_8 with a face F containing at least 5 distinct vertices (the ones to which the noncrossed edges incident to v were joined). By Lemma 2, F contains no more than 5 distinct vertices, and so is pentagonal.

On the other hand, if v has responsibility 2, then there is a crossing c that is not on any edge incident with v. By Lemma 1, none of the edges of D are crossed more than once, and so the two edges e and e' involved in c do not cross edges incident with v. Thus removing edge e leaves a 2-crossing drawing of $K_9 - e$ with v still having responsibility 2. Again by Lemma 1, no edge incident with v is crossed twice, and so removing v yields an S_2 embedding of $K_8 - e$ with a face F containing at least

6 distinct vertices. By arguments similar to those used in the proof of Lemma 2, it can easily be shown that no S_2 embedding of $K_8 - e$ can have a face with more than 6 sides. Thus F is the requisite hexagonal face with 6 distinct vertices. $\square$

Lemma 3 implies that we can prove Theorem 1 by showing the nonexistence of an S_2 embedding of K_8 with a pentagonal face and the nonexistence of an S_2 embedding of $K_8 - e$ with a hexagonal face containing 6 distinct vertices. It is interesting to note that the existence of embeddings of K_n, $n = 4, 5, 6$, with given face size distributions has been studied for its own sake by Lee and White [5].

Note that arguments like those in the proof of Lemma 2 can be used to show that an embedding of $K_7 - e$ in S_1 has one quadrilateral face and the rest triangles. Given this, we have the following lemma.

Lemma 4. *There is no embedding of $K_7 - e$ in S_1 in which the quadrilateral face is disjoint from any one of the triangular faces.*

Proof. If the missing edge can be added across the quadrilateral face, then the theorem can be quickly proved by using the well-known fact that there is only one congruence class of S_1 embeddings of K_7. On the other hand, if the missing edge cannot be added across the quadrilateral face, then we proceed by contradiction.

To that end, assume there is an S_1 embedding M of $K_7 - e$ with a quadrilateral face Q containing vertices 1, 2, 3, and 4 in cycle order, and a triangular face T containing vertices 5, 6, and 7 which has the property that the missing edge cannot be added across Q, that is, that neither edge 13 nor edge 24 is missing. Clearly then we may assume that the missing edge is 15. $Q \cup \{13\}$ cannot lie in a cellular subcomplex of the map, or removing vertices 1 and 3 will disconnect the graph. Thus removing vertices 1 and 3 produces a noncellular embedding of K_5 in the torus, a contradiction. $\square$

Theorem 2. *No embedding of K_8 on S_2 has a pentagonal face.*

Proof. By way of contradiction, assume M is an embedding of K_8 on S_2 with pentagonal face P. The boundary of P must contain 5 distinct vertices, which we will label 1, 2, 3, 4, and 5. By Lemma 2, all faces of M other than P are triangular. We consider two cases.

Case 1: None of the 5 triangles intersecting P in an edge has all three of its vertices in common with P.

Case 2: At least one of the 5 triangles intersecting P in an edge has all three of its vertices in common with P.

Case 1: Suppose the vertices of P are sequentially labelled in CW order. Let the triangle intersecting P in the edge which joins vertex i to vertex $i + 1$ be denoted by T_i for $1 \leqslant i \leqslant 5$, where the values of $i + 1$ should be reduced mod 5 when necessary (as they should throughout this proof). Clearly $(T_i \cap T_{i+1}) \subset \{1, 2, 3, 4, 5\}$. Each T_i has exactly one vertex $v_i \notin \{1, 2, 3, 4, 5\}$.

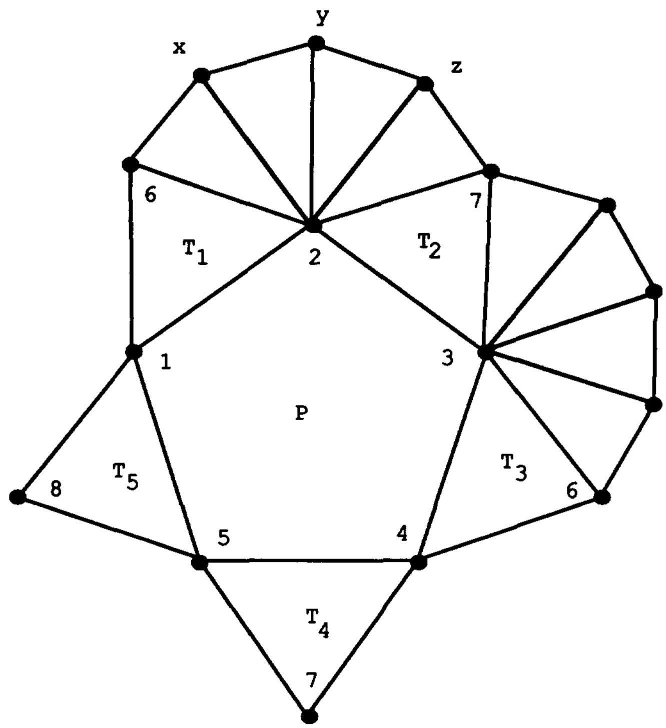

Fig. 2. Case 1 of Theorem 2.

We claim that $v_i \neq v_{i+1}$. If this were not the case, then $T_i \cap T_{i+1}$ would contain the edge joining vertex $i+1$ to vertex v_i, which would force the star complex of vertex $i+1$ to consist only of the three faces, P, T_i, and T_{i+1}. This is impossible, so consequently $v_i \neq v_{i+1}$, and we may assume without loss of generality that $(v_1, v_2, v_3, v_4, v_5) = (6, 7, 6, 7, 8)$ (see Fig. 2).

It follows from the foregoing that 7316 is a CW rotation fragment of 2, so we may write that rotation as $7316xyz$, where $\{x, y, z\} = \{4, 5, 8\}$. If $x = 8$ or $z = 8$ then edges 24 and 25 can be replaced across P to yield an embedding of K_8 which falls under case 2. Thus in this case we may assume that $y = 8$. Likewise we may assume that 8 lies between 1 and 5 in the CW rotation around 3. this implies that edge 85 lies in at least three distinct faces, a contradiction.

Case 2: We assume without loss of generality that T_1 is one of the triangles sharing an edge with P which also has its third vertex in common with P. By reasoning similar to that employed in case 1, the third vertex of T_1 must be 4. Note that a figure similar to Fig. 2 may be helpful in following the subsequent arguments.

Two of the possibilities for the CW rotation around 4 are $(2153xyz)$ and $(21x53yz)$, where $\{x, y, z\} = \{6, 7, 8\}$. In the first alternative, removing vertex 4 and edge 12 produces a noncellular embedding of $K_7 - e$ in S_2. Cutting S_2 along a non-contractible loop in the noncellular region and capping off produces a pentagonal face $23xyz$, in contradiction to Euler's theorem for $K_7 - e$ on the torus.

In the second alternative, removing vertex 4 and edge 12 by the same argument mutatis mutandis yields an embedding of $K_7 - e$ in the torus with triangular face $15x$ and quadrilateral face $23yz$, in contradiction to Lemma 4. The other possibilities for the CW rotation around 4 are essentially the same as those covered, so the case is eliminated and the theorem proved. □

Lemma 5. *If K_8 is embedded on S_2 with two quadrilateral faces which share an edge, then those two faces meet only on that edge.*

Proof. By way of contradiction, assume there is an embedding of K_8 on S_2 in which two quadrilateral faces Q and R meet on an edge, and also meet in a vertex not on that edge. By arguments similar to those used above, it is not possible that $Q \cap R$ contain more than an edge and a vertex not on an edge. Thus we may assume without loss of generality that the hexagonal boundary of $Q \cup R$ contains in CW cyclic order vertices $1, 2, 3, 1, 5, 6$ where $Q \cap R = \{36, 1\}$. Then the CW rotation around 1 must have the form $(53x26yz)$ or $(53xy26z)$, where $\{x, y, z\} = \{4, 7, 8\}$. As in the proof of Theorem 2, removing vertex 1 and edge 36 yields a toroidal embedding of $K_7 - e$ which contradicts Lemma 4. □

Theorem 3. *No embedding of K_8 in S_2 has two quadrilateral faces which share an edge.*

Proof. By way of contradiction, assume there is an S_2 embedding M of K_8 in which two quadrilateral regions Q and R share an edge e. By Lemma 5, $Q \cap R = \{e\}$, so we may assume that the hexagonal boundary of $Q \cup R$ contains the vertices 1–6 in cyclic CW order and that vertices 3 and 6 are the endpoints of e. By Lemma 2, all faces other than Q and R are triangular. Let T_i, $i = 1, \ldots, 6$, be defined analogously to the T_i in the proof of Theorem 2. The proof consists in the elimination of 10 cases, which are listed below. It may be helpful to refer to Fig. 3 while reading the discussion of the generation of the cases. Note that we number the cases in the order in which we treat them as opposed to the order in which they are generated.

We refer to the vertex of T_i which is *not* required to be in it by dint of its definition as the *third vertex* of T_i. Let x, y, and z be the third vertices of T_1, T_3, and T_5, respectively. Firstly it is possible that $\{x, y, z\} \subseteq \{7, 8\}$. Since 7 and 8 are indistinguishable for our purposes, this possibility yields

Case 1: $8 \in T_1$, $8 \in T_3$, $8 \in T_5$.

Case 9: $8 \in T_1$, $7 \in T_3$, $8 \in T_5$.

Case 10: $7 \in T_1$, $8 \in T_3$, $8 \in T_5$.

Secondly, it is possible that only two of x, y, z are in $\{7, 8\}$. If it is $x \notin \{7, 8\}$, then $x \in \{4, 5\}$. These two possibilities for x are indistinguishable for our purposes, so we assume $x = 5$. Then either $y = z$ or $y \neq z$, yielding

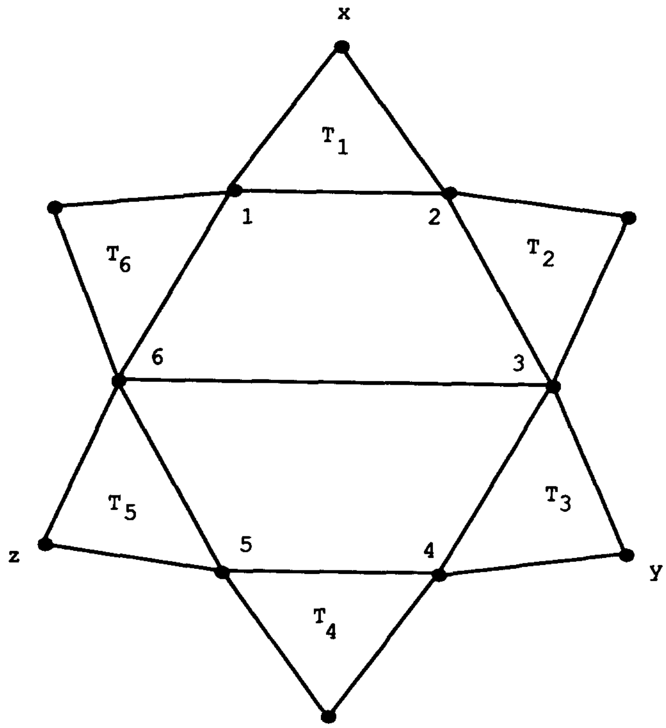

Fig. 3.

Case 2: $5 \in T_1$, $8 \in T_3$, $8 \in T_5$.

Case 7: $5 \in T_1$, $7 \in T_3$, $8 \in T_5$.

On the other hand, the two possibilities $y \notin \{7,8\}$ or $z \notin \{7,8\}$ are indistinguishable, so we assume $z \notin \{7,8\}$. The only possibility is that $z = 2$, and, again, either $x = y$ or $x \neq y$, yielding

Case 3: $8 \in T_1$, $8 \in T_3$, $2 \in T_5$.

Case 8: $8 \in T_1$, $7 \in T_3$, $2 \in T_5$.

Thirdly, if only one of x, y, z is in $\{7,8\}$, then by symmetry we may assume either $x = 8$ or $y = 8$, and that decision constrains the other values, yielding

Case 4: $8 \in T_1$, $1 \in T_3$, $2 \in T_5$.

Case 5: $4 \in T_1$, $8 \in T_3$, $2 \in T_5$.

Note that $5 \in T_1$ is not possible in case 5 because $(5,2,6)$ and $(5,2,1)$ cannot both bound CW triangles in an orientable surface. Henceforth this type of occurrence will be referred to as the *nonorientable reason*, abbreviated NOR. Finally, if $\{x,y,z\} \cap \{7,8\} = \emptyset$ we have

Case 6: $4 \in T_1$, $1 \in T_3$, $2 \in T_5$.

Case 1: Note that throughout the consideration of this case we refer to Fig. 4, which represents the star complex of $Q \cup R$.

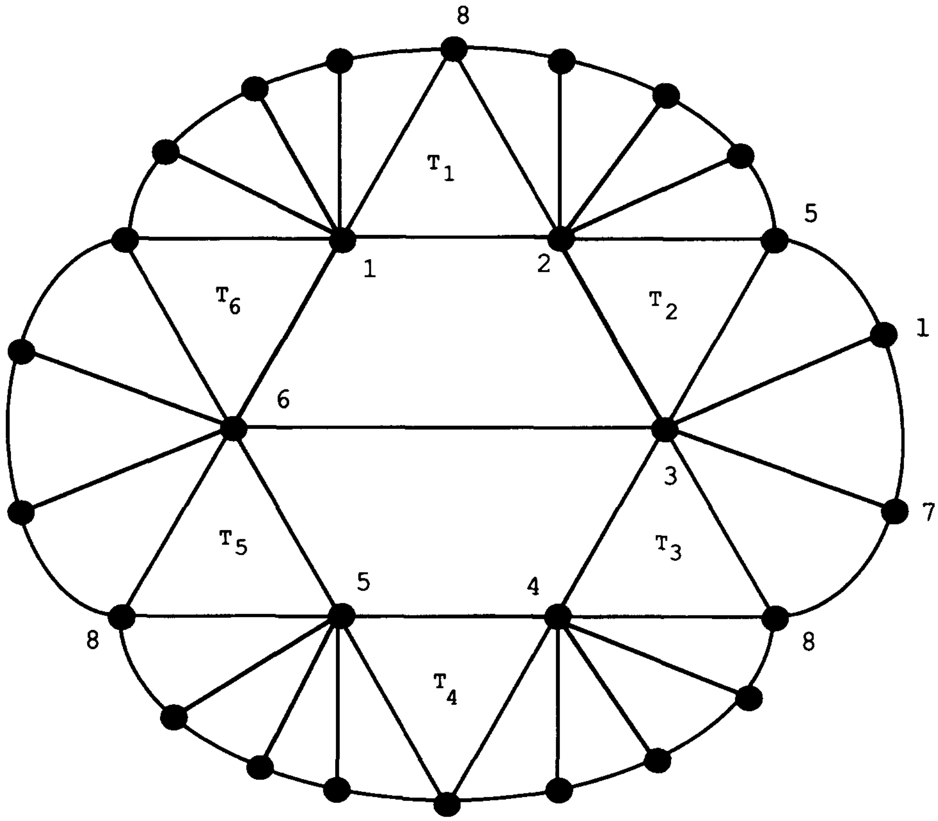

Fig. 4. Case 1 of Theorem 3.

In the CCW rotation around 3, vertices 1, 5, and 7 remain to be added. Neither 1 nor 5 can follow 8 by the NOR; see CCW triangles (8, 1,3), (8, 1, 2) and (8, 5, 3), (8, 5, 6). Thus 7 must follow 8. Vertex $1 \notin T_2$, so $5 \in T_2$. Thus 132 is a CW rotation fragment of 5. Note that 864 is also a CW rotation fragment of 5. Thus the only possibilities for the CW rotation at 5 are (8641327) and (8647132). The first is impossible by the NOR; see CW triangles (7, 8, 5), (7, 8, 3). The second is impossible because it contradicts the already established rotation at 2.

Case 2: Throughout the consideration of this case, we refer to Fig. 5. Similar figures are helpful in the consideration of the other cases, and to encourage the reader to construct them, we provide Fig. 6 as a blank. In the CW rotation around 3, vertices 1, 5, and 7 remain to be added. Neither 1 nor 5 can follow 2 since they are already placed elsewhere in its rotation (henceforth this type of occurrence will be referred to as the *adjacency reason*, abbreviated AR). Thus 7 must follow 2 CW. Vertex 5 cannot precede 8 for the NOR; see CW triangles (1, 5, 3) and (1, 5, 2). Thus 5 must follow 7, and 1 must follow 5. Now, in the CW rotation around 5, vertices 1, 2, 3, and 7 remain to be added.

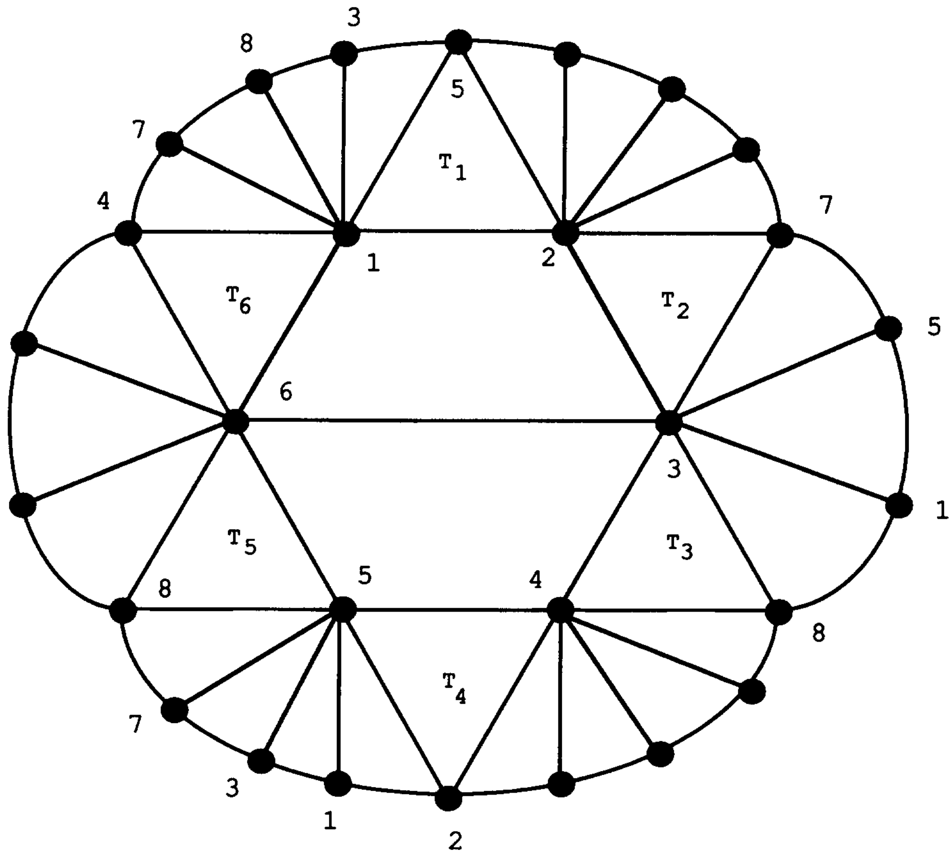

Fig. 5. Case 2 of Theorem 3.

The rotation at 3 implies 137 is a CW rotation fragment at 5, and the rotations at 1 and 2 imply that 21 is also. Thus the CW rotation at 5 is constrained to be (8642137). Now the rotation at 3 constrains the position of 3 and 8 in the rotation at 1, and consequently forces $4 \in T_6$ and constrains 7. This yields CW triangles $(7,8,1)$ and $(7,8,5)$, so the case is eliminated by the NOR.

Case 3: Now, CW around 3 there are three possible locations for vertex 1. By the AR, 1 cannot follow 2, and by the NOR, see CW triangles $(1,8,3)$ and $(1,8,2)$, vertex 1 cannot precede 8. Thus its position is constrained. In the rotation around 2, vertices 4, 5, 6, and 7 remain to be placed. Neither 4 nor 5 can follow 8 by the NOR; see CW triangles $(8,4,2)$, $(8,4,3)$ and $(5,2,6)$, $(5,2,8)$. Thus either 6 follows 8 or 7 follows 8. We treat these as subcases (i) and (ii).

Subcase (i): CW around 6, 8 follows 2 and 5 precedes it. Vertices 4 and 7 remain to be placed in the rotation at 2. Vertex 4 cannot precede 3 CW by the AR, so it must follow vertex 5. This implies $2 \in T_4$, which is impossible by the AR.

Subcase (ii): CW around 2, vertices 4, 5, and 6 remain to be filled in. Vertex 5 cannot follow 7 for the NOR; see CW triangles $(5,2,7)$ and $(5,2,6)$. If vertex 6 were to follow

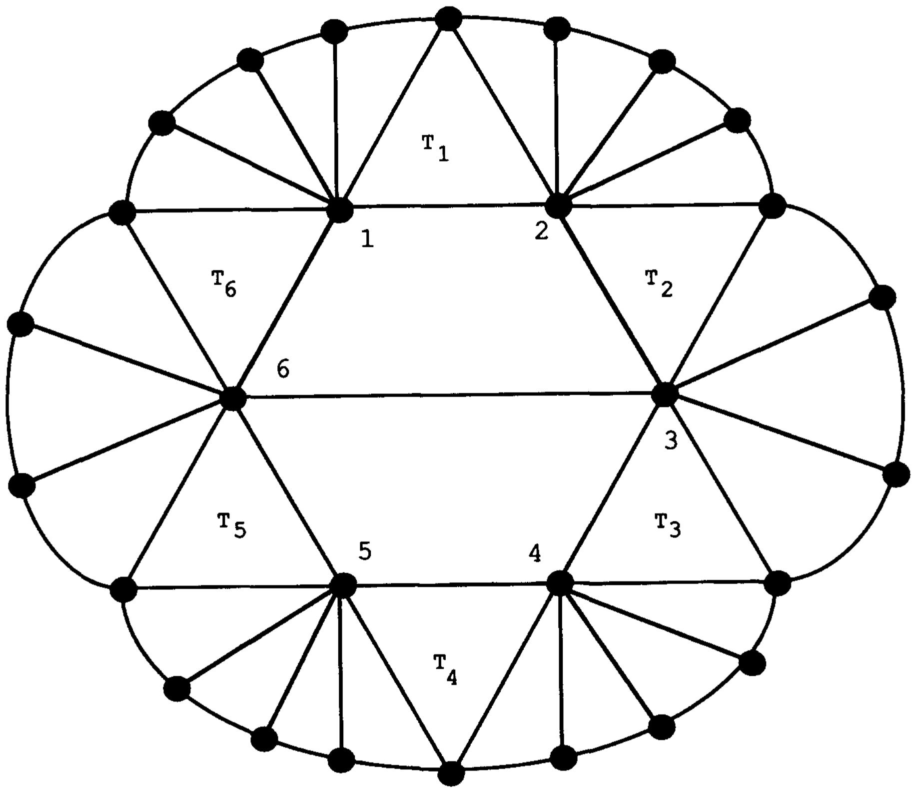

Fig. 6. Generic pattern for cases of Theorem 3.

vertex 7 CW in the rotation at vertex 2, then the rotation at 6 would imply that 5 followed 6 at 2, forcing 4 to come last. This is not possible by the AR. Thus 4 must follow 7. Now, the rotation around 6 implies that around vertex 2, vertices 6 and 5 follow 4 in that order CW. Around 5 the locations of 3 and 1 are constrained, and the location of 7 is constrained around 3. There are two consecutive blank spots in the rotation around 5, which must contain vertices 7 and 8. However, edge 78 is already in two faces, neither of which contains 5, so this subcase too is ruled out.

Case 4: There are three subcases, depending on whether the CW rotation at 1 is ($826x43y$), ($826xy43$), or ($82643xy$) where $\{x, y\} = \{5, 7\}$. We treat only the first two, which are representative.

Subcase (i): Here $x = 7$ since $x = 5$ contradicts the rotation at 4. Note that the CW rotation at 5 must be (2648137) since the other possibility contradicts the rotation at 3. This implies that the CW rotation at 3 is (1462875), which contradicts the rotation at 2.

Subcase (ii): Here $x = 5$ contradicts the rotation at 6, whereas $x = 7$ forces $1 \in T_4$, which is impossible by the AR.

Case 5: In the CW rotation around 3 we need to add vertices 1, 5, and 7. Vertex 1 cannot follow 2 for the AR, and so either 5 or 7 does. We treat these possibilities as subcases (i) and (ii).

Subcase (i): If 5 follows 2 then the position of 3 around 5 is constrained, which in turn constrains the position of 6 around 2. In the CW rotation around 3, either 1 or 7 follows 5. We treat these two possibilities as subsubcases (a) and (b).

Subsubcase (a): The position of 1 around 5 is constrained, and we need to add vertices 7 and 8. The AR implies that CCW we must have 87. This, however, is ruled out by the NOR; see CW triangles $(7, 8, 5)$ and $(7, 8, 3)$.

Subsubcase (b): Here the position of 7 around 5 is constrained. We still need to add 8 and 1 to the rotation at 5. CCW 8 must precede 1 by the AR. However, this is not possible by the NOR; see CW triangles $(1, 8, 5)$ and $(1, 8, 3)$.

Subcase (ii): As above, either 1 or 5 can follow 7 CW around 3, yielding subsubcases (a) and (b).

Subsubcase (a): The rotations already filled in imply that 138 is a CCW rotation fragment at 5. Vertex 1 must follow vertex 2 by the AR. However, that puts edge 12 in three faces, which is not possible.

Subsubcase (b): Here 731 must be a CCW rotation fragment around 5. Vertex 8 cannot precede 4 CCW by the AR, so the complete CCW rotation at 5 must be (8731462). From the rotations already filled in we can deduce that CCW around 4, 2 must follow 1. Vertices 6 and 7 still need to be filled in around 4. Vertex 7 cannot follow vertex 2 CCW since that would contradict the rotation at 2. Thus the CCW rotation at 4 is (1267835). This fact constrains the position of 4 and 7 around 6, which constrains the position of 8 there as well. This yields a contradiction by putting edge 78 in three faces.

Case 6: This case is immediately ruled out by the NOR; see CW triangles $(1, 4, 2)$ and $(1, 4, 3)$.

Case 7: Since reflection about a horizontal axis is a symmetry of Fig. 6, triangles T_2, T_4, and T_6 must fall into one of the four cases not yet ruled out. All three of cases 8, 9, and 10 can immediately be seen to be impossible by the AR, so triangles T_2, T_4, and T_6 must fall into case 7 as well as triangles T_1, T_3, and T_5. This fact implies that $8 \in T_2$, $7 \in T_6$, and either $1 \in T_4$ or $2 \in T_4$. The NOR implies that $1 \notin T_4$; see CW triangles $(1, 5, 2)$ and $(1, 5, 4)$. Thus $2 \in T_4$. The existing rotations constrain the positions of 4 around 2 and 1 around 5. Vertices 3 and 7 remain to be filled in around 5. If 3 follows 8 CCW, then CW around 3 we would have 857, which contradicts the already established rotation at 3. On the other hand, the NOR implies it is not possible to have 73 as a CCW rotation fragment at 5; see CW triangles $(3, 7, 5)$ and $(3, 7, 4)$. Hence this case is eliminated.

Case 8: As before, triangles T_2, T_4, and T_6 must fall into one of cases 8, 9, or 10. Also as before, the AR immediately rules out cases 9 and 10. By the AR, $8 \in T_4$ and $7 \in T_6$. In T_2 we can have either 5 or 6. We will only treat the case where $5 \in T_2$ since the other is practically identical. The existing rotations constrain the position of 6 around 2 and of 3 around 5. Around 3 vertices 1 and 8 remain to be added. The

NOR rules out 18 as a CW rotation fragment at 3; see CW triangles (1, 8, 3) and (1, 8, 2). However, 81 as a CW rotation fragment at 3 contradicts the existing rotation at 5, so this case is eliminated.

Cases 9 *and* 10: As above, triangles T_2, T_4, and T_6 must fall into one of cases 9 or 10. As above, both options are immediately ruled out by the AR. □

Theorem 4. *There is no embedding of $K_8 - e$ in S_2 containing a hexagonal face with 6 distinct vertices.*

Proof. By way of contradiction, assume M is an S_2 embedding of $K_8 - e$ which has a hexagonal face H containing 6 distinct vertices. Theorems 2 and 3 imply that not both ends of the missing edge lie on H. Thus we distinguish two cases: (1) the missing edge has neither end on H, and (2) the missing edge has one end on H.

Case 1: Let the vertices in the hexagon be cyclically labelled 1 through 6 CW. Thus the missing edge is 78. We use the notation T_i for the six triangular faces which intersect the hexagon in an edge as before. It is clearly true that in the rotations of the vertices on the hexagon, 7 must not be consecutive with 8. Furthermore, in these rotations, 7 must be separated from 8 by exactly two vertices. For otherwise, let x be the vertex on the hexagon whose rotation violates this condition. Then one possibility is that vertex 7 is separated from 8 by exactly one vertex, in which case let y be such that $7y8$ is a rotation fragment of x. Then xy can be deleted and added across the hexagon, and then 78 can be added to produce one of the two types of embeddings of K_8 in S_2 forbidden by Theorems 2 and 3. The other possibility is that 7 is separated from 8 by three vertices, say y, z, and w. Then $\{y, z, w\} \subseteq (\{1, 2, 3, 4, 5, 6\} - \{x\})$, so the three edges xy, xz, and xw can be deleted and added across the hexagon. Subsequently, as above, 78 can be added to produce an embedding of K_8 in S_2 of the type forbidden by Theorem 2.

Now consider the star complex of the hexagonal face. By the above considerations in reference to the rotation at 1, either $7 \in T_6$, $8 \in T_6$, $7 \in T_1$, or $8 \in T_1$. These are indistinguishable for our purpose, so without loss of generality we may therefore assume that $7 \in T_1$ (see Fig. 7). This constrains the position of 8 around 1 and around 2, which constrains the position of 7 around 6. This constrains the position of 8 around 6 and 5, which constrains the position of 7 around 5. This constrains the position of 8, and thus 7, around 4. Clearly there is now no way to add 8 to the rotation around 2, which is a contradiction. Thus this case is ruled out.

Case 2: We assume without loss of generality that the missing edge is 17. We have subcases A and B depending on whether $1 \in T_3$ or T_4 or else not.

Subcase A: Without loss of generality, we assume $1 \in T_4$. Then CW around 1 we must have 54 as a rotation fragment, and $5 \notin T_6$ by AR. Thus there are two possibilities for the position of 54, yielding subcases (i) and (ii).

Subcase (i) (*see Fig.* 8(a)): Clearly $3 \notin T_1$ by AR, so the positions of 3 and 8 in the rotation around 1 are constrained. Since 17 is the missing edge, removing vertex 1 and edge 54, and then cutting and capping the noncellular region thus produced yields an

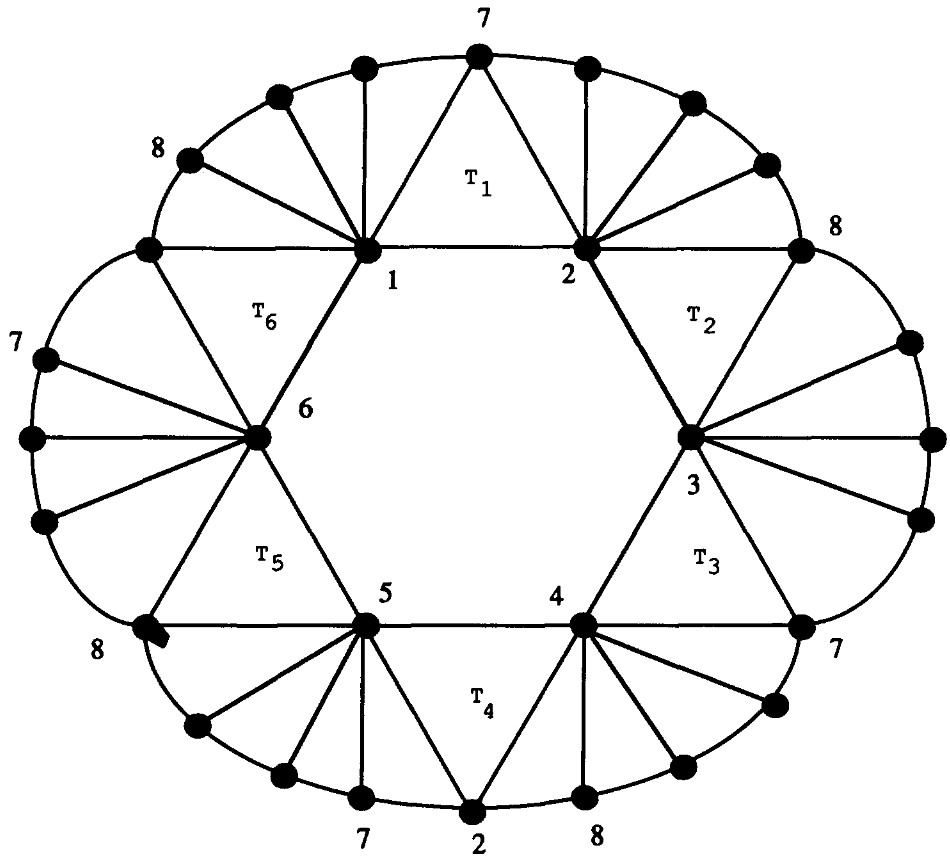

Fig. 7. Case 1 of Theorem 4.

embedding of $K_7 - e$ in the torus with triangular face 5635 and quadrilateral face 48234 (Fig. 8(b)). Since edge 42 is not missing, and cannot lie in a cellular subcomplex with face 48234, removing vertices 2 and 4 yields a noncellular embeddings of K_5 in S_1, which is a contradiction.

Subcase (ii): Around vertex 1 we can have either 83 or 38 as a CW rotation fragment. If we have 83, then the same type of argument used above will yield an embedding of $K_7 - e$ in S_1 with triangular face 4234 and quadrilateral face 53865. As above, but removing vertices 5 and 8 instead of 4 and 2, we obtain an impossible noncellular embedding of K_5 in S_1. Likewise, if the CW rotation fragment at 1 is 38 we obtain a toroidal embedding of $K_7 - e$ with triangular face 4234 and quadrilateral face 58365. From this situation, we can obtain the usual sort of contradiction.

Subcase B: Here $1 \notin T_3$ and $1 \notin T_4$. Thus in the rotation around 1, 3 and 4 are not consecutive, and, likewise, neither are 4 and 5. Note that in the CW rotation around 1, 3 either follows 6, or is three places after 6; for if this is not the case, then either one of

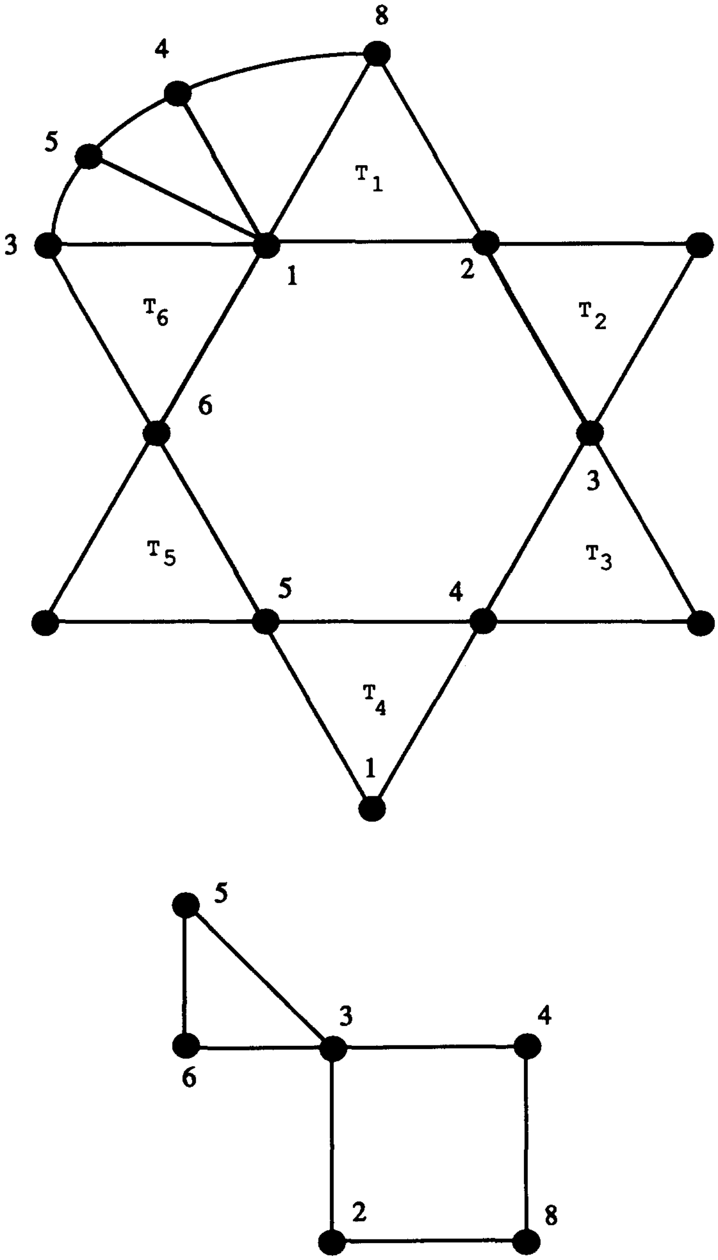

Fig. 8. Subcase 2A of Theorem 4.

the disallowed consecutivities will occur, or else 3 will have to be in two different positions in the rotation around 2. Those two possibilities yield subcases (i) and (ii).

Subcase (i) (*see Fig.* 9): Now, CW around 1, 5 must follow 3, or else one of the forbidden consecutivities will occur; thus the positions of 8 and 4 around 1 are

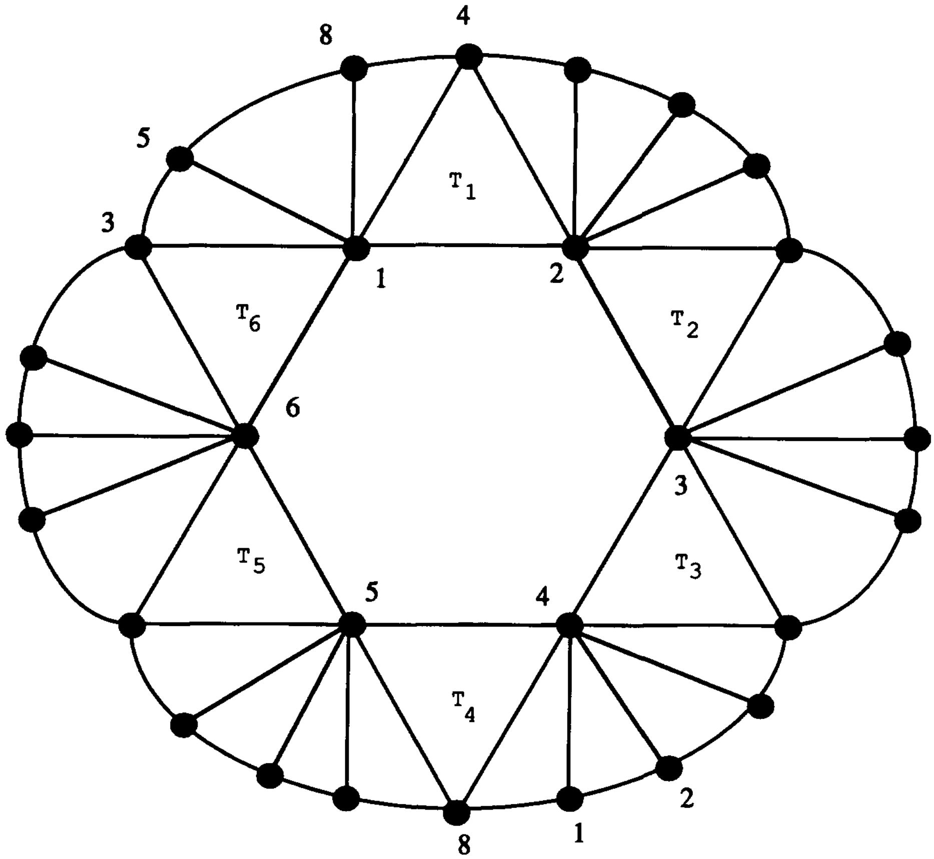

Fig. 9. Subcase 2B of Theorem 4.

constrained as well. Now, 218 must be a CW rotation fragment around 4. Clearly $2 \notin T_3$, and thus either $8 \in T_4$ or else 8 immediately precedes the third vertex of T_4 CW. The second possibility cannot actually occur, for if it did, 7 would be separated from 1 by only one vertex in the rotation at 4, yielding as above (in case 1) one of the forbidden S_2 embeddings of K_8. Thus the position of the CW fragment 218 in the CW rotation at 4 is constrained. Then the position of 8 in the rotation around 1 requires 1 to follow 8 in the CW rotation around 5, which is not possible by the AR.

Subcase (ii): In this case, the positions of 4, 5, and 8 in the rotation around 1 are constrained by the forbidden adjacencies. The sequence 518 must be a CW rotation fragment around 3. The AR implies that $5 \notin T_2$, so either there is one vertex between 2 and 5 CW or else there are two. If there is one then the rotation around 4, which must have 618 as a CCW fragment, would require 8 to be adjacent to the third vertex of T_3 twice, which is not possible. If, on the other hand, there are two vertices CW between 2 and 5 then the same CCW rotation fragment around 4 would require 1 to appear in two different places in the rotation around 8. □

As noted above (in Lemma 3), the combined force of Theorems 2 and 4 proves Theorem 1.

4. Conclusions

Given the fact that Kainen's lower bound is already too small for K_9 and S_2 it seems unlikely that, at least for complete graphs, equality holds for other than the small number of cases which Kainen lists in [4]. The nature of the results in our Section 3, which establish the nonexistence of certain embeddings which are consistent with Euler's formula, seem to hint that the inestimable theorem will be inadequate to the task of calculating the crossing numbers of complete graphs. Finally, the study of face-size distributions of graph embeddings (which seems to be almost completely unexplored except for the aforementioned [5]) promises to be an attractive if difficult field of research.

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References

[1] R. Guy, A combinatorial problem, Bull. Malayan Math. Soc. 7 (1960) 68–72.
[2] R. Guy and A. Hill, The crossing number of the complement of a circuit, Discrete Math. 5 (1973) 335–344.
[3] R. Guy, T. Jenkyns and J. Schaer, The toroidal crossing number of the complete graph, J. Combin. Theory 4 (1968) 376–390.
[4] P. Kainen, A lower bound for crossing numbers of graphs with applications to K_n, $K_{p,q}$, and Q_d, J. Combin. Theory 12 Ser. B (1972) 287–298.
[5] S. Lee and A. White, Random topological graph theory, preprint; Proc. 2nd Internat. Conf. in Graph Theory, Combinatorics, Algorithms and Applications, to appear.