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TYPE 3 DIMINIMAL MAPS ON THE TORUS

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ABSTRACT

A polyhedral map on the torus is diminimal if either shrinking or removing an edge yields a nonpolyhedral map. We show that all such maps on the torus fall into one of two classes, type 2 and type 3, and show that there are exactly two type 3 ones, which are given explicitly.

1. Introduction

A polyhedral map on a surface is called **diminimal** if either shrinking or removing an edge yields a nonpolyhedral map. Steinitz determined that the tetrahedron is the only diminimal map on the sphere, and used this result to establish his famous autonymous theorem [6]. More recently these maps have been studied by D.W. Barnette, who determined the seven diminimal maps on the projective plane [2], and by the author, who determined the one diminimal map on the pinched torus [4] (also known as the pinched sphere and the spindle surface).

The natural next stage of the investigation is to find the diminimal maps on the torus. A complete solution to this problem seems to be far off at this point, although some progress is being made. In this paper, we show that all diminimal maps on the torus are partitioned into two types, called type 2 and type 3. We also classify all type 3 diminimal toroidal maps, of which there are two. Regarding the type 2 diminimal maps, it is known [5] that there are finitely many, but not much besides. The author knows of over 20 such maps, and there are probably many more than that. See [5] for results and conjectures concerning the type 2 diminimal toroidal maps.

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2. Definitions

In this paper, graphs have no loops, multiple edges, or vertices of degree less than 3. A **map** is a 2-cell embedding of such a graph into a surface M. The **faces** of the map are the connected components of $M - G$. Note that we often purposely confuse faces with their bounding circuits. As an example of this, we define a **polyhedral** map to be one with the property that two faces meet, if at all, on a vertex or an edge only. Two faces which meet in such a way are said to meet **properly**.

The edge between vertices x and y is denoted xy. **Shrinking** edge xy means contracting it to a vertex and coalescing any created multiple edges, whereas **removing** edge xy means clipping it out of the graph and coalescing any created 2-valent vertices into the edges in which they lie. Note that these two operations are dual. The inverse operations are called **vertex splitting** and **face splitting** respectively. An edge is called **shrinkable** or **removable** resp. if shrinking or removing it yields a polyhedral map. A polyhedral map with no shrinkable or removable edges is called **diminimal**.

An **obstacle** to the shrinking of an edge is a pair of faces which meet improperly after the edge is shrunk, whereas an **obstacle** to the removal of an edge f is a face which improperly meets the new face created upon the removal of edge f. A **cellular subcomplex** of a map is a set of faces whose union is homeomorphic to a disc. A **3-chain** is a set of three faces of the map in which each intersects the other two. If the three faces have a vertex in common, the 3-chain is said to be **trivial**. A 3-chain is called **planar** if it is contained in some cellular subcomplex. An obstacle $\{A, B\}$ to the shrinking of edge f is called **planar** if $A \cup B \cup \{f\}$ lies in a cellular subcomplex. Similarly, an obstacle A to the removal of an edge f is called **planar** if A along with the two faces containing f lie in a cellular subcomplex of the map. If an edge has a planar obstacle to removing (shrinking) it, it is called **metaremovable (metashrinkable)** for reasons that will be made clear below.

We often abbreviate the phrase "disjoint homotopic nonplanar" by **dhn** as in "dhn circuits". An **annular decomposition** of a polyhedral map on the torus is a set of two or more dhn circuits in the map. An annular decomposition is called **finest** if it is maximal among all such decompositions with respect to the number of dhn circuits it contains. A polyhedral map on the torus is said to be **of type** k if it has k dhn circuits in a finest annular decomposition. Finally, a W_v **circuit** in a polyhedral map is a simple circuit whose intersection with each face is connected.

3. Preliminary Lemmas

We will need the following Lemmas of the author [4]:

LEMMA 1: *Let F and G be the two faces containing edge xy in a polyhedral map M on some surface S. If M is not the tetrahedron and M is minimal with respect to the shrinking of edges, then a pair of faces A and B form an obstacle to the shrinking of edge xy iff x is in one of $A, B; y$ is in the other, and both A, B, and F, and A, B, and G form nontrivial 3-chains.*

LEMMA 2: *In a nontetrahedral map minimal with respect to edge removal, a face F is an obstacle to the removal of edge e iff F lies in a nontrivial 3-chain with the two faces containing e.*

And the following lemmas and theorem of Barnette's ([2], [1], and [3], resp.), the first of which is restated slightly to harmonize with our terminology:

LEMMA 3: *A diminimal polyhedral map has no planar obstacles.*

THEOREM 1: *Every polyhedral map on the torus has a nonplanar W_v circuit.*

LEMMA 4: *The dual of a polyhedral map on a torus is polyhedral.*

Note that in [2], Lemma 3 is proved for diminimal maps on the projective plane, but that the topology of that surface is used nowhere in the proof.

LEMMA 5: *There are no type 1 polyhedral maps on the torus.*

Proof: Let M be a type k polyhedral map on the torus. By Theorem 1 and Lemma 4, its dual has a nonplanar W_v circuit C. As is proved in [1], the boundary of the set of faces of M corresponding to the vertices of a W_v circuit consists of two dhn circuits, so that $k \geq 2$. □

LEMMA 6: *Every diminimal map on the torus is of type 2 or type 3.*

Proof: Due to Lemma 5, we need only show that no diminimal map on the torus is of type k for $k \geq 4$. Suppose there is such a map M. Then M has at least four dhn circuits in some annular decomposition. Let xy be an edge on one of the circuits (see Fig. 1), and let F and G be the two faces containing xy. Note that it is irrelevant whether $F \cap B = \emptyset$ or not, and likewise for $G \cap D$ (see figure).

Let H be an obstacle to the removal of xy. Then due to Lemmas 2 and 3, H must lie in annulus BC or CD, and must lie in a nontrivial, nonplanar 3-chain with F and G. Thus without loss of generality we may assume H lies in annulus BC, and thus has $H \cap G \subset C$ (Fig. 2). In this configuration, however, any obstacle to the shrinking of edge xy will cause two faces to meet improperly. □

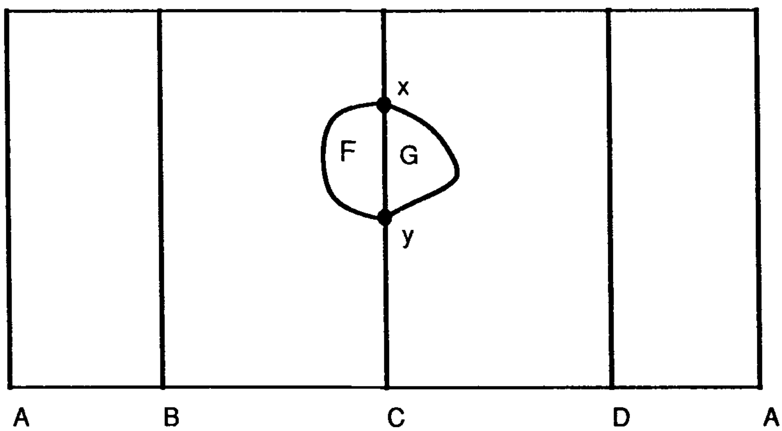

Fig. 1.

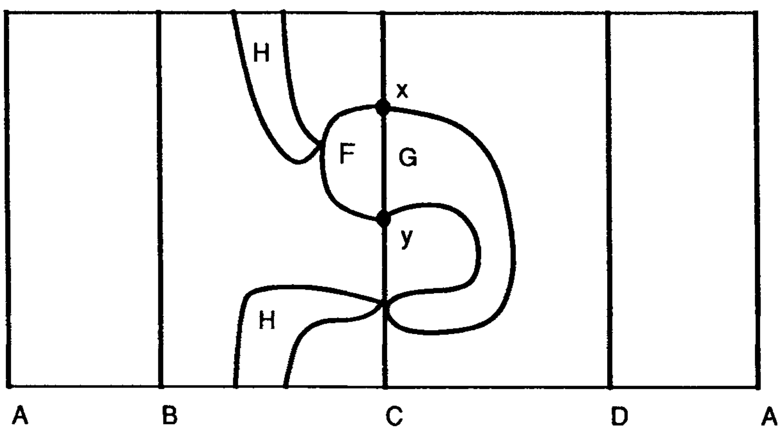

Fig. 2.

4. The main result

In this section, we prove some results specifically relating to type 3 diminimal maps on the torus, and use these to prove the main theorem, which consists of the enumeration and determination of the two such maps. Note that unless explicitly stated otherwise, all lemmas, theorems, and remarks in this section refer to type 3 diminimal maps on the torus.

LEMMA 7: *Any path across one of the three annuli formed by the dhn circuits must consist of a single edge.*

Proof: Suppose there is a path of length ≥ 2 across one of the annuli. Thus there is an edge xy lying in the interior of one of the annuli which has one of its two incident vertices in the interior of the annulus (Fig. 3). Thus, due to Lemmas 2 and 3, any obstacle to the removal of xy must lie entirely within the annulus as well (Fig. 4). In this configuration, however, any obstacle to the shrinking of edge xy would force two faces to meet improperly. □

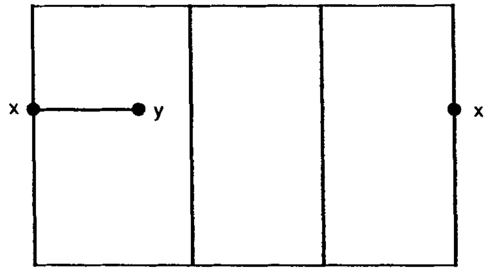

Fig. 3.

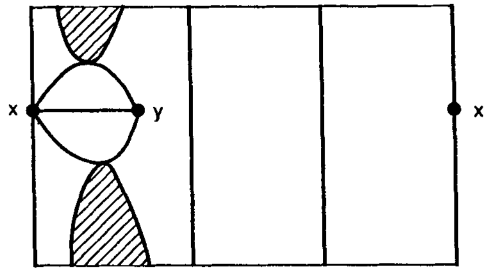

Fig. 4.

We say that two disjoint paths P and Q across an annulus are **consecutive** if one of the two cellular regions into which they divide the annulus contains no paths across the annulus which are disjoint from P or disjoint from Q. Any such region is called a **canton** bounded by P and Q.

LEMMA 8: *A canton must be a face of the map.*

Proof: Due to the annulus on either side, any edges within a canton would either be removable or metaremovable. □

THEOREM 2: *There can be no more than three disjoint edges across an annulus.*

Proof: Assume there are four or more such edges across an annulus, and let four of them be labelled x_1x_2 for $x = a, b, c, d$, such that a_1a_2 and b_1b_2 are consecutive, etc. By Lemma 8, the regions between consecutive edges are faces, and so we call the face between a_1a_2 and b_1b_2 A, etc. Note that there may be more than one face between d_1d_2 and a_1a_2 because, if there are more than four edges across the annulus, those are not consecutive. Finally, let the two annuli not under consideration be labelled I and II (Fig. 5). Now, any obstacle F to the removal of edge b_1b_2 must lie completely in I or completely in II. Without loss of generality, we may assume F lies in II. By Lemmas 2 and 3, F must lie in a nontrivial, nonplanar 3-chain with faces A and B, and there is essentially only one way in which this can occur (Fig. 6).

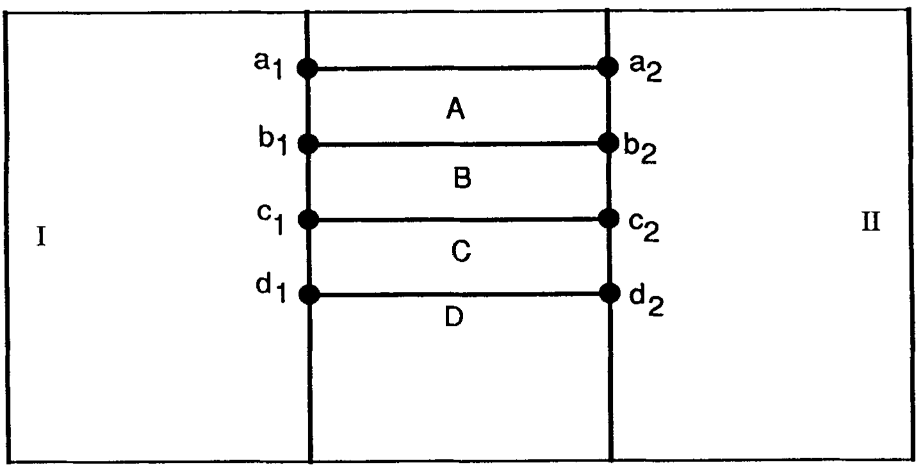

Fig. 5.

Furthermore, any obstacle G to the removal of edge c_1c_2 must lie in a nonplanar, nontrivial 3-chain with faces B and C. Note that it is impossible for F to be an obstacle to the removal of c_1c_2, for if it were, it would lie in a planar nontrivial 3-chain with B and C, contradicting Lemma 3. Note further that if G lay in annulus II, it would be forced to lie in the cellular region determined by F within that annulus. Therefore G must lie in annulus I and make a nontrivial nonplanar 3-chain with faces B and C (Fig. 7). However, by similar arguments, d_1, d_2 is metaremovable, in contradiction to Lemma 3. □

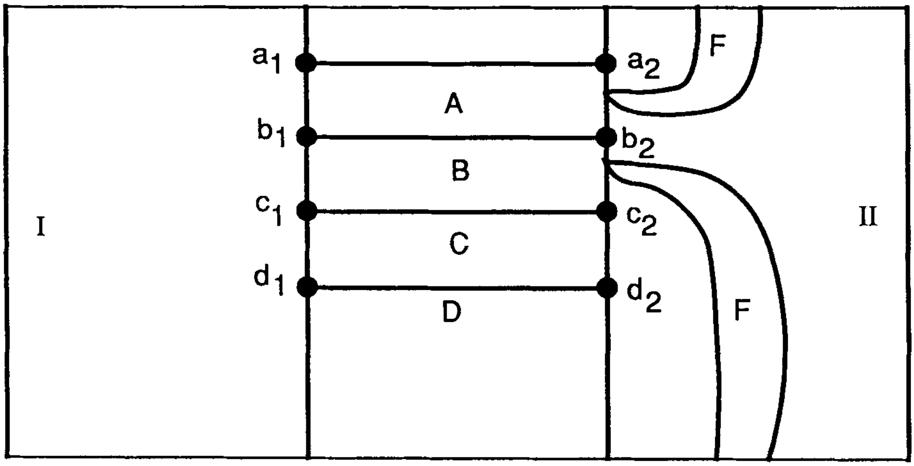

Fig. 6.

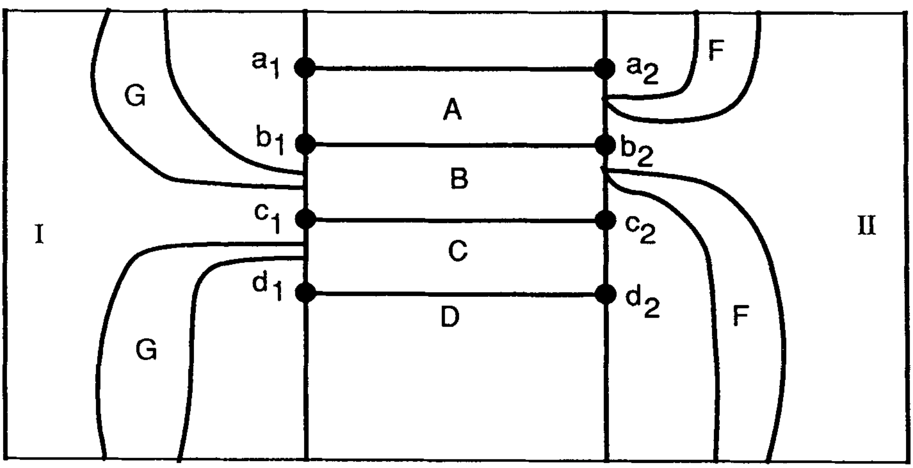

Fig. 7.

Note that polyhedrality, along with Lemma 8, implies that a type 3 diminimal map on the torus has at least 3 edges across each annulus, so we have:

COROLLARY 1: *A type 3 diminimal map on the torus has exactly nine faces, three in each annulus.*

By duality, we have

Corollary 2: *A type 3 diminimal map on the torus has exactly 9 vertices, 3 on each dhn circuit.*

And thus

Corollary 3: *Every type 3 diminimal map on the torus has nine 4-sided faces and nine 4-valent vertices.*

Corollary 4: *There are exactly two type 3 diminimal maps on the torus, shown in Figure 8.*

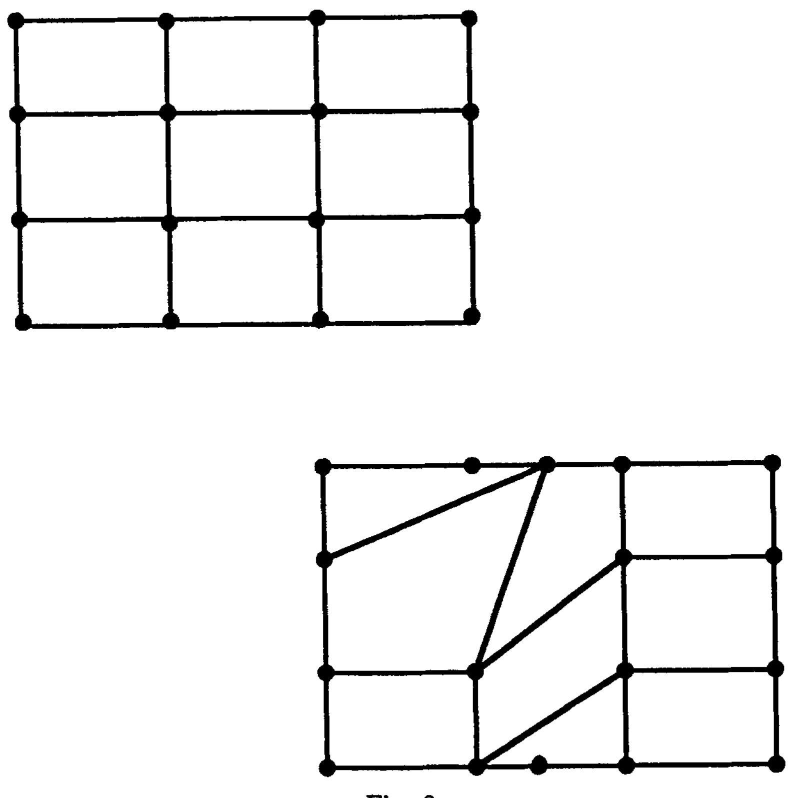

Fig. 8.

Note that these two maps are traditionally known as the triangular picture frame and the twisted triangular picture frame.

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